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LETTER TO THE EDITOR

A self-avoiding walk exponent bound on the thermal Ising exponent on some hierarchical lattices

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Abstract. Calculation of critical exponents on a simple class of hierarchical lattice reveals that $\lambda_s \geqslant \lambda_t$, where λ_s is the self-avoiding walk fixed point eigenvalue and λ_t the Ising thermal eigenvalue. High-dimensional limits of some families of hierarchies obey $\lambda_t \to \lambda_s$ as $D \to \infty$; this convergence replaces the Euclidian concept of upper critical dimension on these lattices. However, families of hierarchies for which $D \to \infty$ but with constant connectivity do not show this convergence.

The existence of hierarchical lattices (Berker and Ostlund 1979, Kaufman and Griffiths 1981, 1982) with exactly known critical exponents prompts the search for relationships between exponents of different but related models. Some time ago Domb (1970, 1972) examined connections between series expansions of Ising and SAW models on regular lattices; SAW graphs are a subset of Ising graphs. Mackenzie (1976) considers the same. In field theoretic formalism both models are members of the n-vector class, de Gennes (1972) and Ma (1976).

The author has previously reported Ising model exponents on hierarchies (Melrose 1983a, b). Here SAW exponents are calculated and comparisons made.

There are many, as yet unclassified, varieties of hierarchical lattices (Kaufman and Griffiths 1981, 1982, McKay *et al* 1982, Melrose 1983c, McKay and Berker 1984). Perhaps the simplest class, those studied below, are *bond hierarchies*: choose any linear graph, the basic cell, on which two sites, the nodes, define a bond decoration; the lattice is generated by an iterative decoration as illustrated in Kaufman and Griffiths (1981, 1982) and Melrose (1983a, b, c). Further conditions placed on the examples studied here are that the hierarchy be fully iterated in that every bond is decorated at each stage and that it be symmetric in that the nodes are equivalent on the basic cell. Some basic cells are shown in figure 1. Melrose (1983b) defines the intrinsic dimension, D, by $D = \log(g)/\log(b)$ and the connectivity, Q, by $Q = \log(q)/\log(b)$, where g is the number of bonds on the basic cell, q is the minimum number of bonds which if cut on the basic cell separate the nodes and b is the number of bonds on the shortest path between the nodes.

Dhar (1978) studied SAWs on his early examples of hierarchical lattices. Shapiro (1978) introduced SAW renormalisation on Euclidian lattices. Recently several authors (Rammal *et al* 1984, Ben-Avraham and Havlin 1984, Klien and Seitz 1984) have studied SAWs on the Sierpinski gasket.

The self-avoiding constraint and ensemble weighting can be applied to walks in diverse ways (Amit *et al* 1983). Here the traditional definition of a SAW is studied:

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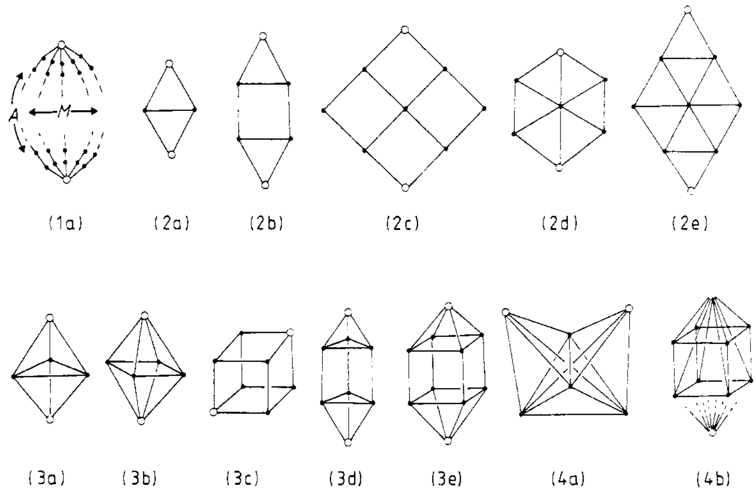

Figure 1. Some basic cells.

each walk graph does not self intersect and ensembles of walks are considered with each graph being given a weight S^N, where N is the number of steps on the walk. It is perhaps not appropriate to consider some actual discrete time walk rather than just an ensemble of walk graphs as stated.

Renormalisation of the SAW model is carried out graphically: the step weight S is renormalised

$$S' = G_c(S) = \sum_N c_N S^N \tag{1}$$

where $G_c(S)$ is the generating function for the numbers of SAWs of N steps, c_N, which cross from one node to the other on the basic cell; by construction renormalisation factors on each basic cell of the hierarchy. The mth iterate of (1), S'_m, is the generating function for walks crossing an mth unit, where an mth unit is the finite hierarchy formed at the mth decoration. For the basic cell (1a) of figure 1, the Migdal–Kadanoff hierarchies, one finds

$$S' = MS^A \tag{2}$$

and on the cell (2b)

$$S' = 2S^3 + 4S^4 + 2S^5. \tag{3}$$

Table 1 gives recursion relations for all cells of figure 1 and families of figure 2. One may easily enumerate the crossing walks by hand.

The recursion relations (1) are polynomial with $S'(S=0)=0$ and a single unstable fixed point $S' = S = S^*$ such that as $m \to \infty$ and for $S < S^*$, $S'_m \to 0$ and $\lambda_m \to 0$, whilst for $S > S^*$, $S'_m \to \infty$ and $\lambda_m \to \infty$, where $\lambda_m = \mathrm{d}S'_m/\mathrm{d}S'_{m-1}$. The ensemble expectation at weight S of the number of steps on walks crossing the basic cell, $\langle n_c \rangle_1$, obeys $\langle n_c \rangle_1 \propto \lambda_1(S)$, and to cross an mth unit

$$\langle n_c \rangle_m \propto \lambda_m \lambda_{m-1} \ldots \lambda_1(S). \tag{4}$$

For $S < S^*$, $\langle n_c \rangle_m \to 0$ as $m \to \infty$; whilst for $S > S^*$, $\langle n_c \rangle_m \to \infty$. At $S = S^*$, $\langle n_c \rangle_m \propto \lambda_s^m$, where λ_s is the fixed point eigenvalue $\mathrm{d}S'/\mathrm{d}S|_{S^*}$ (note this gives explicit physical

Table 1. SAW recursion relations for the examples in the text.

For the cells of figure 1

$$S' = \sum_n c_n S^n,$$

with c_n:

cell \ n	1	2	3	4	5	6	7	8	9
2a	0	2	2	0	0	0	0	0	0
2b	0	0	2	2	2	0	0	0	0
2c	0	0	0	6	0	4	0	2	0
2d	0	1	6	8	4	2	0	0	0
2e	0	0	0	6	22	28	20	6	0
3a	0	3	6	6	0	0	0	0	0
3b	0	4	8	8	8	0	0	0	0
3c	0	0	6	0	6	0	6	0	0
3d	0	0	3	12	24	24	18	0	0
3e	0	0	4	16	32	48	48	48	32
4a	0	4	12	24	24	0	0	0	0
4b	0	8	24	48	96	144	240	192	144

and zero for $n > 9$.

For the families of figure 2

$$\text{Fa:} \quad S' = \sum_{k=1}^{n} \left(S^2 + \sum_{i=1}^{k-1} S^{2+i} + \sum_{j=1}^{n-k} S^{2+j} \right),$$

$$\text{Fb:} \quad S' = \sum_{i=2}^{n+1} \frac{n!}{(n-i+1)!} S^i,$$

$$\text{Fc:} \quad S' = (2s + 4S^2 + 2S^3) \left(\sum_{i=0}^{n} S^{1+i} \right)^2 .$$

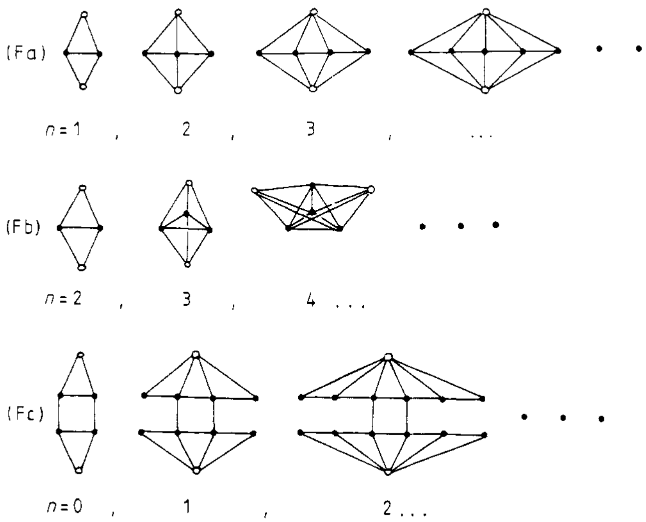

Figure 2. Some families of basic cells.

interpretation of the fixed point eigenvalue and expresses the self similarity of the 'critical' SAW walks). (The physical interpretation of model eigenvalues on hierarchies is a useful bonus; for the Ising model, the renormalised coupling gives interface free energies on units and the internal energy of the interface is proportional to the thermal eigenvalue.) As mth units are of linear scale b^m one may define an intrinsic fractal dimension for the SAWs: $D = Y_s = \log(\lambda_s)/\log(b)$ (intrinsic because distance is defined by the intrinsic lattice metric (Melrose 1983b)). Following Shapiro (1978) one may show that the ensemble correlation length $\mathscr{C}(s)$ diverges as $S \to S^*_-$ with $(S) \sim (S-S)^{-\nu_s}$, where $\nu_s = 1/D_s$; note that the relation $\nu_s = 1/D_s$ proposed on regular lattices by Havlin and Ben-Avraham (1982) is found exact on the hierarchies.

Melrose (1984) discusses renormalisation of random walks on hierarchies and gives similar interpretations of the rational recursion relations found in this case (note the random walk recursion relations are normalised in that they are generating functions for probabilities of crossing units whilst the SAW relations (1) are not).

Attention now turns to numerical results. For the hierarchies of figure 1, table 2 compares values of S^* and λ_s with those of the Ising fixed point J^* and thermal eigenvalue λ_t found by Melrose (1983b). One observes in all cases the bound $\lambda_s > \lambda_t$ as stated in the abstract. Although the author has been unable to prove this bound explicitly it is straightforward to show that $\tanh(J^*) > S^*$.

Table 2. Ising and SAW fixed points and eigenvalues

	2c	2b	2e	3c	2a	3d
J^*	0.7111	0.6673	0.5241	0.4326	0.4407	0.3781
λ_t	2.3352	2.0737	2.4293	2.6096	1.8284	2.8164
S^*	0.5175	0.4783	0.3731	0.3784	0.3660	0.2823
λ_s	4.3765	3.6471	5.0167	3.3166	2.2679	4.0361
	3e	**3a**	**3b**	**2d**	**4a**	**4b**
J^*	0.3184	0.2351	0.1832	0.3269	0.1606	0.0941
λ_t	3.1917	2.1577	2.2173	2.4038	2.3312	2.2764
S^*	0.2822	0.2178	0.1750	0.2628	0.1538	0.0931
λ_s	4.1631	2.4086	2.3561	2.9677	2.4989	2.3120

The Ising recursion relations are developed in $t = \tanh(J)$ via the expectation $\langle\sigma_1\sigma_2\rangle_c$ of the two nodes on the basic cell in a high temperature expansion:

$$t' = \tanh(J') = \langle\sigma_1\sigma_2\rangle_c, \tag{5}$$

where $\langle\ \rangle_c$ denotes an expectation over the basic cell partition function; note that $\langle\sigma_1\sigma_2\rangle_c$ is not the expectation $\langle\sigma_1\sigma_2\rangle$ on the full lattice. (One can view the renormalisation as a many-one mapping between high temperature graphs in $\tanh(J)$ on the original and renormalised lattice; the SAW renormalisation may be viewed similarly.) The recursion relation in t' explicitly involves that of the SAWs (1):

$$t' = (g_1(t) + g_2(t))/Z(t) \tag{6}$$

where $g_1(t)$ is a sum over crossing SAW graphs as in (1), $G_c(S) = g_1(S)$, $g_2(t)$ is a sum over SAW graphs with connected and disconnected closed loops and $Z(t)$ is the partition function of the basic cell, a sum over all connected and disconnected loops. (A loop is a graph with an even number of edges at each site and multiple edges are not allowed in the above graphs.) The bound $t^* > S^*$ follows directly from the bound $S' > t'$. Now $S' > t'$ if

$$g_1(x)(Z(x)-1) > g_2(x). \tag{7}$$

On the left of (7) one has a direct product of SAW graphs and closed loops which includes graphs with multiple bonds. On the right of (7) one again has SAWs with loops, with the exclusion however of multiple bonds, hence clearly term by term the bound is proven. (Note a factor $\cosh^g(J)$ divides out of (6) as is usual in $\tanh(J)$ expansions.)

The bound $\tanh(J^*) > S^*$ was proved on the square lattice by Fisher and Sykes (1959) and the above extends this to the bond hierarchies. Domb (1970, 1972) was similarly unable to prove the exponent bound although it is obeyed by the known regular lattice results and the ε expansion (Ma 1976).

Families of hierarchies which for large cell limits have $D \to \infty$ are easily constructed, Melrose (1983b). Figure 2 shows three such families; family Fa has $D/Q \to 1$ as $D \to \infty$, Fb has $D/Q \to 2$ as $D \to \infty$ and Fc has $Q = \text{constant}$ and hence $D/Q \to \infty$ as $D \to \infty$. A general way of forming a family is to take any basic cell and to consider a multiplicity, M, of this cell connected at the nodes; this is illustrated by the Migdal–Kadanoff hierarchies (1a).

From the MKH recursion relation, (2), one finds $S^* = M^{1/(1-A)}$ and $\lambda_s = A$ giving with $b = A$, $\nu_s = 1\ \forall A, M$; this exponent is that of the 1D lattice and the linear nature of the MKH has been stressed by Melrose (1984). As is well known (Migdal 1975, Melrose 1983a) the Ising model on the MKH obeys $\lambda_t < A$ and $\lambda_t \to A$ as $M, D \to \infty$. Figure 3 shows the eigenvalue variation with M of families formed from multiples of cells (2a) and (3a), again $\lambda_s > \lambda_t$ and $\lambda_t \to \lambda_s \to b$ as $M, D \to \infty$. The Ising and SAW fixed points converge and tend to zero as $M \to \infty$. (Note the peak in the Ising eigenvalue of 2a hints at the complexity of the graph statistics underlying λ_t; that is whilst at high D, SAW statistics are most important, at lower D other, as yet unknown, statistics are significant. A number of cells, 3a, 3d, 3e, 2a, 2b, 2d, 2e, show similar peaks.)

The limiting behaviour $\lambda_s \to b$, of the SAW recursion relation as $M \to \infty$ is shown to be that of the recursion relation of the shortest crossing paths of the form (2). Let

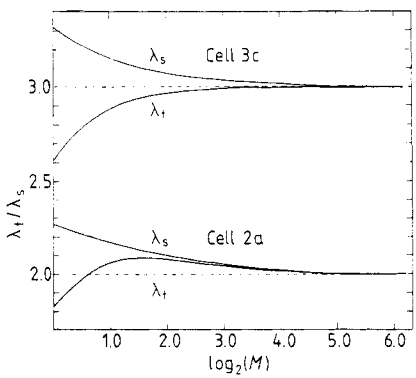

Figure 3. Eigenvalues for multiplicities of cells 3c and 2a.

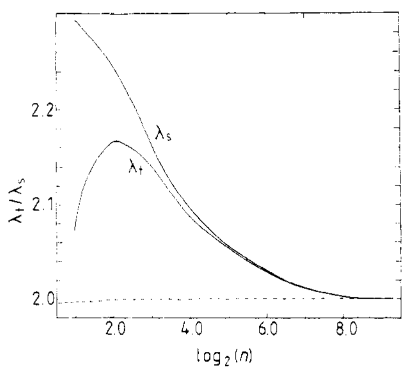

Figure 4. Eigenvalues for the family Fa of figure 2 where n is defined in figure 2.

there be p shortest paths of length b between the nodes of the cell multiply combined, then for M cells

$$S' = M\left(pS^b + \sum_i c_i S^{b+i}\right); \tag{8}$$

note that M enters both SAW and Ising recursion relations as a multiplicative constant. Now substitute in (8) the fixed point $S_0^* = (Mp)^{1/(1-b)}$ of the shortest paths recursion relations $S' = MpS^b$, giving

$$S'(S_0^*) = S_0^* + \sum_i (c_i p^{(b+i)/(1-b)}) M^{1+(b+i)/(1-b)}. \tag{9}$$

Clearly as $M \to \infty$ and for $b > 1$ all terms in the sum tend to zero and $S'(S_0^*) \to S_0^*$; similarly for the eigenvalue $\lambda_s \to b$.

Figures 4 and 5 show eigenvalue variations for the families Fa and Fb respectively. With Fa, $\lambda_t \to \lambda_s \to b = 2$ whilst on Fb, $\lambda_t \to \lambda_s \to 3$ as $D \to \infty$. Clearly Fa tends to the shortest paths result as discussed for multiplicity. The limit of Fb however shows that the Ising results converge to the SAW results in general. In both families fixed points converge and tend to zero as $D \to \infty$. However the results for family Fc shown in figure 6 suggest that the divergence of the connectivity with D is a necessary condition for the Ising and SAW models to converge; the behaviour of this family is quite intuitive given the form of the basic cells.

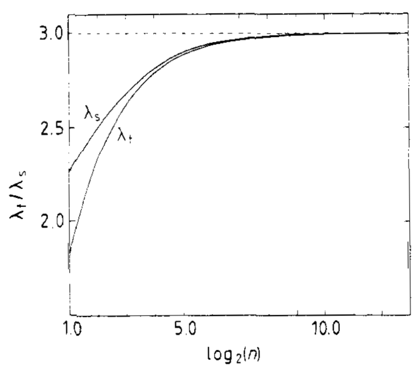

Figure 5. Eigenvalues for the family Fb.

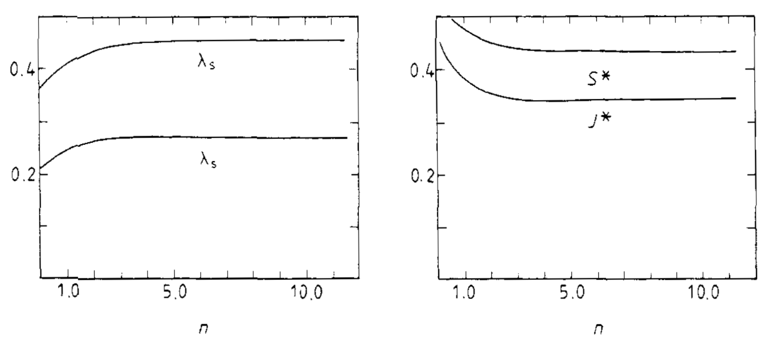

Figure 6. Eigenvalues and fixed points for the family Fc. $\lambda_s \to 4.635$, $\lambda_t \to 2.7216$, $S^* \to 0.3446$, $J^* \to 0.4365$.

Melrose (1984) calculates the spectral dimension, F (Dhar 1977, Rammal and Toulouse 1983), of the above hierarchies. Whilst on fractals it has been proposed (Rammal and Toulouse 1983) that $F = 4$ indicates a crossover of SAW statistics to those of random walks no such behaviour is found for SAWs on the hierarchies here: for example on the MKH $F = D$, yet $\nu_s = 1\ \forall D$.

The $D \to \infty$ convergence shown above replaces the concept of upper critical dimension known on regular lattices. Furthermore the convergence of the Ising model to the SAW model is seen to be more fundamental, in the sense that it extends to the hierarchies, than a coincident convergence of Ising and SAW models to the Gaussian model at $d = 4$ in Euclidian space. Figures 3-5 clearly show that the Ising eigenvalue is influenced by the SAW eigenvalue prior to any convergence of λ_s, rather than both models tending independently to a common limit.

In addition to relationships between ν_{Is} and ν_{SAW}, as considered above, Domb (1970, 1972) also sought relationships between both the SAW exponent γ and the Ising susceptibility exponent, and the analogous closed polygon exponent and the Ising specific heat exponent. The evaluation of these SAW exponents and possible relationships remains for future investigation.

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